The Limits of Coding with Joint Constraints on Detected and Undetected Error Rates

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***Abstract*— We develop a remarkably tight upper bound on the performance of a parameterized family of bounded angle maximum-likelihood (BA-ML) incomplete decoders. The new bound for this class of incomplete decoders is calculated from the code's weight enumerator, and is an extension of Poltyrev-type bounds developed for complete ML decoders. This bound can also be applied to bound the average performance of random code ensembles in terms of an ensemble average weight enumerator.**

We also formulate conditions defining a parameterized family of optimal incomplete decoders, defined to minimize both the total codeword error probability and the undetected error probability for any fixed capability of the decoder to detect errors. We illustrate the gap between optimal and BA-ML incomplete decoding via simulation of a small code.

I. Introduction

Practical coding systems often employ a powerful error-correcting code to combat noise, together with another code used to detect errors only. However, calling one code an error-correcting code and another an error-detecting code is a misnomer, because both types of codes inherently possess both correction and detection capabilities. For any fixed code its relative effectiveness at correcting and detecting errors depends on how it is decoded.

Some codes, such as Reed-Solomon codes decoded by standard algebraic methods, have combined correction and detection capabilities that are amenable to analysis. Various constructions of low-density parity check (LDPC) codes, decoded by iterative message passing with codeword validation, combine capacity-approaching error correction with an ability to detect most uncorrectable errors. But these are just particular solutions to the general problem of determining the tradeoffs that must be made to simultaneously achieve both good error correction and error detection.

Our goal is to simultaneously minimize both the overall codeword error probability P_w and the portion P_u of this error probability that corresponds to undetected errors. In this paper we explore this problem for soft decoding of block codes $\mathcal{C}$ of length n with M equally likely, equal-energy codewords $\mathbf{c}_i, i = 0, ..., M-1$, represented in n-dimensional Euclidean space and received in the presence of additive white Gaussian noise (AWGN). As noted above, the tradeoff of detected and undetected error rates depends on both the code $\mathcal{C}$ and the (generally incomplete) decoder $\mathcal{D}$ used to decode it. A more incomplete decoder can lower the undetected error rate P_u at the expense of increasing the overall word error rate P_w.

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II. Families of Incomplete Decoders

For any code $\mathcal{C}$ and a complete or incomplete decoder $\mathcal{D}$, we define P_d to be the detected error probability (erasure probability of the decoder) and P_u to be the undetected error probability. The total probability of codeword error is then given by $P_w = P_u + P_d$. There are many good bounds on the minimum word error probability P_w^{ML} achievable with maximum-likelihood (ML) decoding. In this paper we wish to characterize fundamental tradeoffs between P_w and its two components P_u, P_d.

We define three interesting families of incomplete decoders:

- A *bounded distance maximum-likelihood* (BD-ML) decoder $\mathcal{D}^{BD}(D_d)$ is characterized by its maximum decoding distance D_d. If the Euclidean distances from the n-dimensional received word $\mathbf{x}$ to each of the codewords, denoted by $D_i = ||\mathbf{x} - \mathbf{c}_i||^2$, are all greater than D_d, then $\mathcal{D}^{BD}(D_d)$ outputs an erasure (i.e., a detected error). Otherwise it outputs the codeword $\mathbf{c}_{i^*(\mathbf{x})}$ closest to $\mathbf{x}$, which is the $i^*(\mathbf{x})$ that minimizes $D_i, i = 0, ..., M-1$.
- A *bounded angle maximum-likelihood* (BA-ML) decoder $\mathcal{D}^{BA}(\theta_d)$ is characterized by its maximum decoding angle θ_d. If the Euclidean angles between the n-dimensional received word $\mathbf{x}$ and each of the codewords, denoted by $\theta_i = \cos^{-1}(\mathbf{x} \cdot \mathbf{c}_i / ||\mathbf{x}||)$, are all greater than θ_d, then $\mathcal{D}^{BA}(\theta_d)$ outputs an erasure (i.e., a detected error). Otherwise it outputs the codeword $\mathbf{c}_{i^*(\mathbf{x})}$ that minimizes $\theta_i, i = 0, ..., M-1$.
- A *bounded reciprocal likelihood ratio maximum-likelihood* (BRLR-ML) decoder $\mathcal{D}^*(\Lambda_d)$ is characterized by a maximum reciprocal likelihood ratio Λ_d. If the ratios of the average likelihood $\bar{p}(\mathbf{x}) = (1/M)\sum_{i=0}^{M-1} p_i(\mathbf{x})$ for receiving $\mathbf{x}$ to the conditional likelihoods $p_i(\mathbf{x})$ given each of the possible codewords, denoted by $\Lambda_i = \bar{p}(\mathbf{x})/p_i(\mathbf{x})$, are all greater than Λ_d, then $\mathcal{D}^*(\Lambda_d)$ outputs an erasure (i.e., a detected error). Otherwise it outputs the codeword $\mathbf{c}_{i^*(\mathbf{x})}$ that minimizes $\Lambda_i, i = 0, ..., M-1$.

Note that our definitions of BD-ML and BA-ML decoders do not restrict them to maximum decoding distances D_d or angles θ_d corresponding to non-overlapping bounded decoding regions; if the bounded decoding regions around two or more codewords overlap, the ML rule is used. This convention allows the BD-ML and BA-ML decoders, as well as the BRLR-ML decoders, to all approach optimal ML performance (minimizing P_w) on the AWGN channel, whenever their decoding limits D_d or θ_d or Λ_d are suitably large. However, these three families differ in their effectiveness of trading off P_w and P_u when their decoding limits are small enough to permit them to detect errors.

The BRLR-ML decoders $\mathcal{D}^*(\Lambda_d)$ are *optimal incomplete decoders* in the sense that for a given code $\mathcal{C}$ they minimize both P_w and P_u for a given decoder erasure probability P_d. This is justified as follows. An optimal incomplete decoder $\mathcal{D}^*$ should follow the ML decoding rule except in its erasure region $\mathcal{E}$. If a differential volume $d\mathbf{x}$ around received word $\mathbf{x}$ were assigned to the erasure region $\mathcal{E}$, the differential increase in P_w would be $dP_w = p_{i^*(\mathbf{x})}(\mathbf{x})\,d\mathbf{x}$. The corresponding increase in P_d would be $dP_d = \bar{p}(\mathbf{x})\,d\mathbf{x}$. An optimal family of incomplete decoders should minimize the increase of P_w with P_d at every possible operating point. This implies that the erasure region for a given P_d should consist of those points $\mathbf{x}$ which minimize dP_w/dP_d, or alternatively maximize $dP_d/dP_w = \bar{p}(\mathbf{x})/p_{i^*(\mathbf{x})}(\mathbf{x}) = \Lambda_{i^*(\mathbf{x})}$. In other words, define $\mathcal{E}(\Lambda_d) = \{\mathbf{x} : \Lambda_{i^*(\mathbf{x})} \leq \Lambda_d\}$, and let the parameter Λ_d vary from 0 to 1. The corresponding parameterized values of P_w and P_d are given by: $P_d(\Lambda_d) = \int_{\mathcal{E}(\Lambda_d)} \bar{p}(\mathbf{x})\,d\mathbf{x}$, and $1 - P_w(\Lambda_d) = \int_{\mathcal{E}^c(\Lambda_d)} p_{i^*(\mathbf{x})}(\mathbf{x})\,d\mathbf{x}$ where $\mathcal{E}^c(\Lambda_d)$ is the complement of $\mathcal{E}(\Lambda_d)$.

Figure 1 gives a comparison of the tradeoffs among P_u, P_d, and P_w for the $(8,4)$ extended Hamming code with BD-ML, BA-ML and optimal incomplete decoding. These results were obtained by simulation.

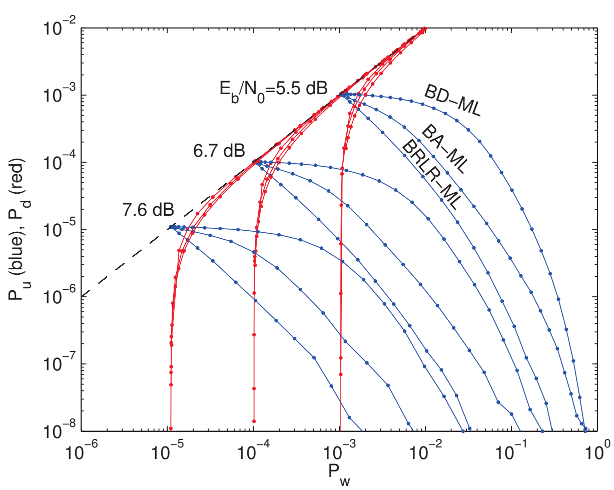

Fig. 1. Comparison of P_u, P_d, P_w tradeoff curves for the $(8,4)$ extended Hamming code with BD-ML, BA-ML and optimal incomplete decoding.

The tradeoff of most interest is P_u versus P_w, as shown by the blue curves. There are three clusters of curves, obtained at three values of AWGN signal-to-noise ratio E_b/N_0 as indicated. The three values of E_b/N_0 were chosen to produce word error probabilities P_w^{ML} under ML decoding around 10^{-3}, 10^{-4}, and 10^{-5}. Within each cluster are three curves, corresponding to three families of decoders. From left to right in each cluster, they are the optimal incomplete (BRLR-ML) decoders, the BA-ML decoders, and the BD-ML decoders.

The blue curves are standard tradeoff curves, where the lower left corner of the plot is the unachievable region for a given code and decoder at a given E_b/N_0. Their negative slope shows that one can achieve a lower P_u only at the price of a higher P_w, and vice versa. There are limits in both directions. P_w cannot be made smaller than the P_w achieved by the ML decoder (at which point $P_u = P_w$). In the other direction, there is the trivial limit that $P_u = 0$ only if $P_w = 1$, so the right edge of this graph is a vertical asymptote for all the blue curves.

The curves in Fig. 1 show significant differences in the P_u versus P_w tradeoff for the various decoder families, even though all of them approach the same ML performance when their decoding limits are large enough. Optimal incomplete decoders achieve the best tradeoff, of course, and BA-ML decoders outperform BD-ML decoders at least for this code.

The red curves show the decoders' erasure or detection probabilities, $P_d = P_w - P_u$. Note that P_d is zero for the ML decoder, and otherwise P_d is nearly equal to P_w. The detected error probability P_d is generally of less interest than the undetected error probability P_u, and so the remainder of this paper will focus on the P_u versus P_w (blue) tradeoff curves.

III. Performance Analysis for BA-ML Decoders

In the remainder of this paper we restrict our attention to a performance analysis of BA-ML decoders, which are more amenable to analytical bounds and approximations than optimal incomplete decoders. With BA-ML decoding, ML word error probability P_w^{ML} is achieved as $\theta_d \to \pi$, and in fact for all decoding angles $\theta_d \geq \theta_{\max}$, where $\theta_{\max}$ is the angle of the smallest cone *circumscribing* the true codeword's ML Voronoi region. For such values of θ_d, $P_w = P_u = P_w^{ML}$ and $P_d = 0$, and their approach to these ML limits as depicted in Fig.1 is obtained by evaluating the performance of BA-ML decoders $\mathcal{D}^{BA}(\theta_d)$ for decoding angles θ_d approaching $\theta_{\max}$ from below. For any $\theta_d \leq \theta_{\min}$, where $\theta_{\min}$ is the angle of the largest cone *inscribed in* the true codeword's ML Voronoi region, the decoding cones of the incomplete decoder $\mathcal{D}^{BA}(\theta_d)$ are non-overlapping, and evaluation of the probability of correct decoding, $P_c = 1 - P_w$, simplifies to the probability that the received word falls within the decoding cone of angle θ_d around the true codeword. For $\theta_d = \theta_{\min}$ the word error probability P_w reaches that of a traditional *bounded angle* decoder, $P_w^{BA} > P_w^{ML}$. Further reduction of P_w from P_w^{BA} toward P_w^{ML} is obtained for larger decoding angles, $\theta_{\min} < \theta_d < \theta_{\max}$, where evaluation of P_w and P_u is more complicated.

To analyze BA-ML decoders we follow the general approach outlined in [4], which for ML decoders includes the union bound, the sphere packing lower bound, a Poltyrev-type upper bound [2], [3], and many more complicated lower and upper bounds as special cases. See also [5], [6], [7], [8], and references therein, for discussions of various good bounding techniques applicable to ML decoding.

The approach in [4] follows Shannon [1] and considers the differentially thin conical shell $d\mathrm{S}_n(\theta)$ between two circular cones of half-angles θ and $\theta + d\theta$, each with vertex at the origin and axis passing through the correct codeword $\mathbf{c}_0$, as shown in Figure 2. This shell contains a fraction

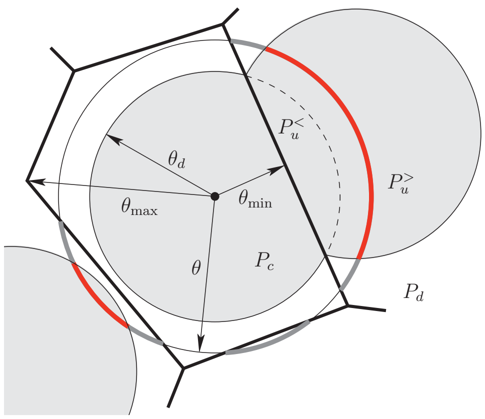

Fig. 2. Illustration of the conical angle geometry underlying our analysis of BA-ML decoding. Contributions to $F(\theta)$ are shown in wide arcs; contributions to $F(\theta;\theta_d)$ consist only of the segments shown in red.

$d\Omega_n(\theta) = \frac{n-1}{n}\frac{\Gamma(n/2+1)}{\Gamma(\frac{n+1}{2})\sqrt{\pi}}(\sin\theta)^{n-2}d\theta$ of the total solid angle in n-dimensional space. For a given code, a certain fraction of the shell's solid angle (a) falls outside the true codeword's ML Voronoi region, or, equivalently, (b) belongs to a different codeword's ML Voronoi region. This fraction is called the *code geometry function* $F(\theta)$. It depends on the geometry of the code, but not on the Gaussian noise distribution. The exact ML word error probability P_w^{ML} can in principle be evaluated in terms of $F(\theta)$ as

$$P_w^{ML} = \int_0^{\pi} F(\theta)p_0(\theta)\, d\theta$$

where $p_0(\theta)\, d\theta$ is the probability that the noise from the channel deflects the received word into the differential shell at angle θ from the true codeword $\mathbf{c}_0$.

Various upper and lower bounds on P_w^{ML} were obtained in [4] by evaluating this integral with various upper and lower bounds on $F(\theta)$ substituted for the true $F(\theta)$. Here we consider only a Poltyrev-type upper bound derived from a first-order upper bound $F^+(\theta)$ similar to a union bound based on the code's weight enumerator A_w:

$$P_w^{ML} \le \int_0^{\pi} F^+(\theta)p_0(\theta)\, d\theta$$

where

$$F(\theta) \le F^+(\theta) \triangleq \min\left[1, \sum_{w>0} A_w \int_0^{\beta_w(\theta)} d\Omega_{n-1}(\theta_1)\right],$$

where

$$\beta_w(\theta) = \underline{\cos}^{-1}\left(\tan\psi_w \cot\theta\right)$$

and

$$\psi_w = \sin^{-1}\left(\sqrt{w/n}\right)$$

is half the angle between the true codeword and a codeword at Hamming distance w from it. Here $\underline{\cos}^{-1}(\cdot) \triangleq \cos^{-1}(\min[1, \max[-1, \cdot]])$ denotes the inverse cosine function applied to a clipped argument.

In our present application to incomplete BA-ML decoders, the conditions (a) or (b) defining $F(\theta)$ are no longer equivalent. We find it convenient to define a code geometry function $F(\theta;\theta_d)$ for BA-ML decoders to be the fraction of the conical shell at angle θ from the true codeword that (c) falls within the BA-ML decoding region of an incorrect codeword. This $F(\theta;\theta_d)$ is directly useful for computing the *undetected* error probability of BA-ML decoders.

If a BA-ML decoder $D^{BA}(\theta_d)$ is used instead of the ML decoder $D^{ML} = D^{BA}(\theta_{\max})$, then it is convenient to evaluate $P_w(\theta_d)$, $P_u(\theta_d)$, $P_d(\theta_d)$, each as a sum of two components, one resulting from received words with angle $\theta \le \theta_d$ and the second from received words with angle $\theta > \theta_d$, where θ is the angle between the received word and the true transmitted codeword. For this decomposition, it is convenient to deal temporarily with the probability of correct decoding, $P_c(\theta_d) = 1 - P_w(\theta_d)$, instead of $P_w(\theta_d)$, and to introduce $P_{tot}^<(\theta_d) \triangleq \int_0^{\theta_d} p_0(\theta)\, d\theta$ and $P_{tot}^>(\theta_d) \triangleq \int_{\theta_d}^{\pi} p_0(\theta)\, d\theta$ as the total probabilities that the received word lands in these two regions. These total probabilities decompose as follows:

$$P_{tot}^<(\theta_d) = P_c(\theta_d) + P_u^<(\theta_d)$$

$$P_{tot}^>(\theta_d) = P_d(\theta_d) + P_u^>(\theta_d)$$

where the "<" and ">" components of $P_u(\theta_d)$ are contributed by received words within the two regions. Refer again to Fig. 2 for a Venn-diagram depiction of the four components $P_c(\theta_d)$, $P_d(\theta_d)$, $P_u^<(\theta_d)$, and $P_u^>(\theta_d)$ in the conical geometry. We note that for a BA-ML decoder, there are no contributions to $P_c(\theta_d)$ for $\theta > \theta_d$, and no contributions to $P_d(\theta_d)$ for $\theta \le \theta_d$. Thus only $P_u(\theta)$ is decomposed into "<" and ">" components. Finally, noting that the (unit) probability of the entire space of received words decomposes alternatively as $P_{tot}^<(\theta_d) + P_{tot}^>(\theta_d) = 1 = P_c(\theta_d) + P_w(\theta_d)$, we rewrite the first of these equations in terms of $P_w(\theta_d)$ rather than $P_c(\theta_d)$:

$$P_w(\theta_d) = P_{tot}^>(\theta_d) + P_u^<(\theta_d)$$

The total undetected error probability is determined from the sum of its two components:

$$P_u(\theta_d) = P_u^<(\theta_d) + P_u^>(\theta_d)$$

We find it convenient to first evaluate $P_{tot}^>(\theta_d)$, then bound or approximate the two components of the undetected error probability, $P_u^<(\theta_d)$ and $P_u^>(\theta_d)$, and finally use the last three equations to calculate the corresponding bounds or approximations to $P_u(\theta_d)$, $P_w(\theta_d)$, and $P_d(\theta_d)$.

A. Bounds on $P_u^<(\theta_d)$ and $P_u^>(\theta_d)$

As with our earlier bounds on the error probability of ML decoders, it is useful to express the two components of the undetected error probability as integrals with respect to the channel probability measure of a code geometry function $F(\theta;\theta_d)$ defined for BA-ML decoders as the fraction of a differentially thin conical shell at angle θ that is decoded to an incorrect codeword (not erased):

$$P_u^<(\theta_d) = \int_0^{\theta_d} F(\theta;\theta_d)\, dP(\theta)$$

$$P_u^>(\theta_d) = \int_{\theta_d}^{\pi} F(\theta;\theta_d)\, dP(\theta)$$

First we note that $F(\theta;\theta_d) = 0$ for all $\theta \le \theta_d$ if $\theta_d \le \theta_{\min}$, and $P_u^<(\theta_d) = 0$ in this case. Thus, non-trivial bounds or approximations to $P_u^<(\theta_d)$ are only needed when $\theta_d > \theta_{\min}$, i.e., when the bounded-angle decoding cones overlap one another.

The BA-ML decoder simply narrows the decoding region for the true codeword by applying an additional condition $\theta \le \theta_d$ beyond the "nearest codeword" condition applied by the ML decoder. Since any received word contributing to the undetected error probability of an ML decoder must necessarily be at least as close to an incorrect codeword as to the true codeword, the code geometry function $F(\theta;\theta_d)$ defined for a BA-ML decoder is equivalent to that defined previously for the ML decoder for all received word angles $\theta \le \theta_d$, i.e.,

$$F(\theta;\theta_d) = F(\theta) \text{ for all } \theta \le \theta_d$$

Thus, the corresponding bound on $P_u^<(\theta_d)$ is a straightforward extension of the Poltyrev-type upper bound derived previously for ML decoders, with only a trivial restriction of the interval of integration.

$$P_u^<(\theta_d) \le \int_0^{\theta_d} F^+(\theta) p_0(\theta)\, d\theta$$

The bound on $P_u^>(\theta_d)$ requires a small generalization when a BA-ML decoder is used in place of an ML decoder. Again we are interested in evaluating a Poltyrev-type upper bound obtained from an upper bound on $F(\theta;\theta_d)$ based on the code's weight enumerator A_w:

$$P_u^>(\theta_d) \le \int_{\theta_d}^{\pi} F^+(\theta;\theta_d) p_0(\theta)\, d\theta$$

where $F^+(\theta;\theta_d)$ is a first-order upper bound to $F(\theta;\theta_d)$ given by

$$F(\theta;\theta_d) \le F^+(\theta;\theta_d) \triangleq \min\left[1, \sum_{w>0} A_w \int_0^{\beta_w(\theta;\theta_d)} d\Omega_{n-1}(\theta_1)\right]$$

with

$$\beta_w(\theta;\theta_d) = \overline{\cos}^{-1}\left(\frac{\cos(\min[\theta,\theta_d]) - \cos\theta\cos 2\psi_w}{\sin\theta \sin 2\psi_w}\right)$$

This is a generalization of the corresponding upper bound obtained in [4] for ML decoders. The $\cos(\min[\theta,\theta_d])$ term in the expression for $\beta_w(\theta;\theta_d)$ simplifies to $\cos\theta_d$ for all values of $\theta > \theta_d$ needed in the evaluation of the upper bound on $P_u^>(\theta_d)$, or to $\cos\theta$ for all values of $\theta \le \theta_d$ needed in the evaluation of the upper bound on $P_u^<(\theta_d)$. In the latter case, the argument of $\overline{\cos}^{-1}(\cdot)$ reduces to $\tan\psi_w \cot\theta$, the same expression used previously to define $\beta_w(\theta)$ for ML decoding. Thus, the more general term $\cos(\min[\theta,\theta_d])$ is convenient because it can be used to bound either $P_u^<(\theta_d)$ or $P_u^>(\theta_d)$.

Figure 3 compares the Poltyrev-type upper bounds obtained from the code geometry function $F(\theta;\theta_d)$ with the results in Fig. 1 obtained by simulation for the $(8,4)$ extended Hamming code with BA-ML decoding. Also shown for comparison in this figure is the tradeoff curve obtained from the bound for the (24,12) extended Golay code operating at an E_b/N_0 producing an ML word error probability close to that of the middle curves for the extended Hamming code.

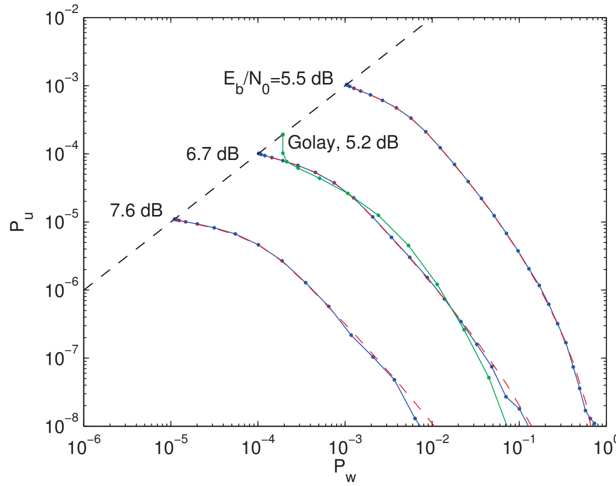

Fig. 3. Comparison of bounds and simulations of the P_u, P_w tradeoff curves for the $(8,4)$ extended Hamming code with BA-ML decoding, at values of E_b/N_0 selected to produce ML word error probabilities of about 10^{-3}, 10^{-4}, and 10^{-5}. Also shown for comparison is the same P_u, P_w tradeoff under BA-ML decoding for the $(24,12)$ extended Golay code operating at an E_b/N_0 producing an ML word error probability of about 10^{-4}.

Fig. 3 illustrates the phenomenally close approximation to true BA-ML performance obtained from our Poltyrev-type bound, as we have extended it to the family of incomplete BA-ML decoders. In this figure the bound is virtually indistinguishable from the curve obtained by simulation. It is also interesting that the P_u versus P_w tradeoff curve for the extended Golay code (obtained from the bound) appears very similar to that of the much smaller extended Hamming code, as long as the channel operating points (E_b/N_0) for the two codes are selected to yield similar ML performance. This similarity might be connected to the fact that both the Hamming and Golay codes are nearly perfect, or it might have wider applicability (a subject for further research).

B. Bounds for Random Code Ensembles

Both $P_u^<(\theta_d)$ and $P_u^>(\theta)$ are linear functionals of their respective code geometry functions, and the equations for determining $P_u(\theta_d)$, $P_w(\theta_d)$, $P_d(\theta_d)$ from $P_u^<(\theta_d)$, $P_u^>(\theta)$ are linear also. Thus, if a specific code $\mathcal{C}$ is replaced by a random ensemble of codes, the bounds derived for specific codes remain valid for the ensemble averages of these error probabilities, $\bar{P}_u^<(\theta_d)$, $\bar{P}_u^>(\theta)$, $\bar{P}_w(\theta_d)$, $\bar{P}_d(\theta_d)$, if the corresponding bounds on $F(\theta;\theta_d)$ are replaced by their ensemble averages.

A completely random code ensemble is a special case for which bounds are not needed, because the ensemble averages can be computed exactly. In this case, the ensemble average code geometry function is evaluated as

$$\bar{F}(\theta;\theta_d) = 1 - [1 - \Omega_n(\min[\theta,\theta_d])]^{M-1}$$

because $1 - \bar{F}(\theta;\theta_d)$ is simply the probability that none of $M-1$ randomly selected alternatives to the true codeword falls closer in angle to the received word $\mathbf{x}$ than: (a) maximum angle θ if $\theta \leq \theta_d$, or (b) maximum angle θ_d if $\theta > \theta_d$. The corresponding ensemble average error probabilities for the random code ensemble are evaluated as:

$$\bar{P}_w(\theta_d) = 1 - \int_0^{\theta_d} [1 - \Omega_n(\theta)]^{M-1} p_0(\theta)\, d\theta$$

$$\bar{P}_u(\theta_d) = 1 - \int_0^{\pi} [1 - \Omega_n(\min[\theta,\theta_d])]^{M-1} p_0(\theta)\, d\theta$$

$$\bar{P}_d(\theta_d) = [1 - \Omega_n(\theta_d)]^{M-1} \int_{\theta_d}^{\pi} p_0(\theta)\, d\theta$$

Figure 4 plots tradeoff curves showing ensemble average $\bar{P}_u(\theta_d)$ versus $\bar{P}_w(\theta_d)$ for BA-ML decoding of random codes of length $n = 8$ and dimension $k = 4$. The tradeoff curves in Fig. 4 are considerably worse than those in Fig. 1 or Fig. 3 for the nearly perfect $(8,4)$ extended Hamming code. This indicates that a completely random code ensemble does not provide a very useful estimate of the achievable BA-ML performance tradeoffs for such small values of n and k.

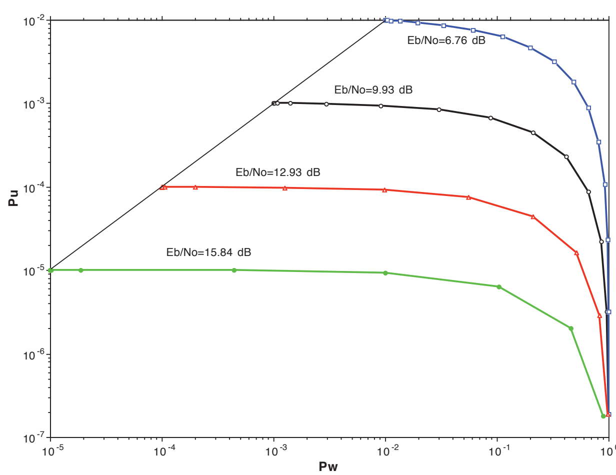

Fig. 4. Tradeoff curves showing ensemble average $\bar{P}_u(\theta_d)$ versus $\bar{P}_w(\theta_d)$ for BA-ML decoding of random codes of length $n = 8$ and dimension $k = 4$.

IV. Conclusion

We have taken a preliminary look, via simulations and bounds, at the inherent tradeoffs between a code's ability to correct and detect errors, depending on the type of decoder used. We looked at three parameterized families of incomplete decoders, all of which approach ML performance in the limit of complete decoding when they lose any capability for error detection. Optimal incomplete decoders were defined to yield the best P_u versus P_w tradeoff curve, but performance bounds for this family of decoders are intractable and simulations are only practical for small codes. The BA-ML decoders constitute a sub-optimal family for which extremely tight Poltyrev-type upper bounds on performance were obtained. Such bounds also constitute (looser) upper bounds on the performance of optimal incomplete decoders.

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